Int. J. Solids Structures, 1971, Vol. 7, pp. 843 to 861. Pergamon Press. Printed in Great Britain

MULTI-LOBED INFLATED MEMBRANES: THEIR STABILITY UNDER FINITE DEFORMATION

J. D. C. CRISP and L. J. HART-SMITH

Department of Mechanical Engineering, Monash University, Clayton, Victoria 3168, Australia

Abstract—The stability of inflated membraneous spheres of rubber-like materials is examined theoretically and stability criteria are confirmed by experiment. A negative pressure–volume gradient is found to be a critical condition for instability. Another condition derives from geometric considerations and usually involves a critical volume enclosed within the entire system. These two conditions together determine the stability of any multi-lobed inflatable membraneous structure, regardless of material. Equilibrium states of two and three inter-connected spherical balloons, whether identical or not, are presented with corresponding stability diagrams and the inflation history of such configurations is adduced. The scope of poly-stable states is described and confirmed experimentally. Specific results are presented for rubber-like materials that are described by the exponential–hyperbolic elasticity parameters developed elsewhere [1]. Inadequacies of other theories [3–11] are traced either to improper elastic representations or to the neglect of one of the dual conditions for stability.

1. INTRODUCTION

A NEW representation of the elasto-mechanical properties of rubber-like materials in terms of exponential–hyperbolic elasticity parameters has been developed [1], and it has been verified [2] by a study of static deformations. Here it is applied to problems in static instability which have not hitherto been adequately explained using earlier representations.

Knowledge of the existence of instabilities in inflated rubber-like membranes is widespread, and attempts have been made to account for these theoretically. The first such analysis seems to have been that of Mallock [3] in 1891 who predicted instability for both spheres and freely extending circular cylinders upon the attainment of a maximum pressure at diametral extension ratios of 1·73 and 1·82, respectively. In pressure-controlled experiments on india-rubber specimens Mallock reproduced the cylindrical instability and, it seems, inferred that of the sphere but noted a "close apparent agreement between the observed and theoretical results (which is) somewhat illusory". His theoretical result derives from a nonlinear elasticity formulation which exhibits an anisotropy of which, apparently, he was unaware. Subsequently, Osborne and Sutherland† [4] in 1909 predicted the stability of spherical balloons of linearly elastic material but noted that the absence of a pressure maximum conflicted with their volume-controlled experimental observations of rubber balloons. Accordingly, they introduced the nonlinear "elastic law of O. Frank" successfully to achieve a qualitative reconciliation; the peak pressure is then predicted to occur at a diametral extension ratio of 1·50. Further, they interconnected two identical spherical balloons and observed a duality of stable configurations, and a snap-through instability which allowed the interchange of the sizes of the balloons, for certain conditions

† These authors mention concurrent work of R. du Bois-Reymond which, however, is not available to the present authors.

of the contained volume. They offered no explanation, however. Much later, in 1948, Kubo [5] studied the characteristics of and predicted a snap-through instability for a sphere of neo-Hookean (Treloar) material, upon the attainment of a peak pressure at a diametral extension ratio of 1·38. He claimed complete stability, on the other hand, for circular cylindrical membranes of the same hypothetical material. Levinson [6], and later Johnson and Soden [7], reproduced Kubo's result exactly for the neo-Hookean sphere although the former omitted to identify any constraint on pressure or volume at the onset of instability. Levinson [8], in addition, found a sphere composed of a Mooney material to be unstable or not according to the existence or absence of a stationary (maximum) value of the pressure with respect to distension; in other words, according to the actual elastic constants used in the Mooney formulation. The corresponding cylinders he reckoned to be always stable.

A feature common to all of these studies is the identification of instability with the entrance upon a distension region having a pressure gradient that is negative with respect to one or other finite extension. This criterion is generally offered with an intuitive or physical basis and only occasionally with analytical support. Burton [9], in discussing the distension of biological vessels, for example, deduces such a criterion by physical reasoning based on a graphical display of the nonlinear elasticity of a hypothetical tissue material. He later misuses [10] the criterion in demonstrating the instability ("blow-out") of a linearly elastic cylinder under plane strain which by his own analysis, is actually stable. In any event, it is generally not recognized, it seems, and only rarely stated that such a criterion is relevant only to a pressure-controlled condition (and, more precisely, the criterion then refers to a pressure gradient with respect to enclosed *volume*). This *was* recognized by Panovko and Gubanova [11] who, after correctly criticizing the analysis by Rzhanitsyn [12] of the finite deformation of an incompressible linear elastic spherical membrane, reproduced his value, $\sqrt{e} \approx 1{\cdot}65$, of the diametral extension ratio at which peak pressure is attained; and went on carefully to distinguish between the stability of equilibrium when the shell is connected to an infinite reservoir, on the one hand, and inflated by a pump on the other. The former corresponds to instability under constant pressure, the latter to complete stability under constant volume. In predicting constant-pressure instabilities arising from a negative pressure gradient, however, it is necessary to give attention to the structural boundary conditions: not all boundary conditions allow the existence of a peak pressure, quite apart from whether the class of material permits it. Johnson and Soden [7] revealed this in their paper, referred to earlier, which includes a study of the circular cylinder.

The more general problem of instabilities under varying pressure and/or volume was examined by Corneliussen and Shield [13] who studied comprehensively the vibration and stability of the simultaneously inflated and extended membraneous circular cylinder. They discovered instabilities arising from axial or circumferential compression, but more interestingly, instability in a region of large extension ratios. Their criterion for this unstable regime involves the deformation and loading states of course (and, therefore, the boundary conditions), and the initial aspect ratio of the cylinder as well as the gradients of both the pressure and axially applied force with respect to the distending and stretching extension ratios. They delineated, in detail, the stability boundaries for Mooney and Neo-Hookean material formulations. Their predictions, which are general, have yet to be demonstrated, however, in real cylindrical membranes composed of rubbery materials.

Overall, the published work is in some respects contradictory and presents neither a clear nor a comprehensive picture of the stability of inflated membranes of *real* rubber-like materials. The use (at *large* extension ratios) of inadequate representations of the elasto-mechanical properties of rubber-like material has, naturally, precluded the reconciliation of theoretical analyses and experiments where this has been attempted.

The pressure-gradient criterion of stability has not always been accompanied by an understanding of the need to state explicitly the corresponding boundary conditions. No attempt has been made to relate the behaviour of rubber membranes to that of soap films which is well documented and dates back to the work of Plateau [14]. Indeed, the unstable behaviour of soap films is shown here to lie within the same general framework of that of membranes composed of materials, such as rubber, that do not have constant surface stress-resultants. (The stability of soap films is reviewed in the Appendix.)

It is the purpose here to elucidate the actual quantitative behaviour of real membranes. The problem of instability under constant pressure is given no further attention: sufficient conditions are the existence of a negative pressure gradient with respect to volume, together with the absence of any restriction on the volume of distending fluid communicating with the membraneous lobe or lobes. A comprehensive set of stability criteria are, therefore, developed concerning the stability of equilibrium states, under the conditions of *volume constancy*, of all membranes regardless of the material composition and without restriction as to the number of discrete lobes. It is assumed that the distending fluid is essentially incompressible so that when inflation is achieved with air (or other gas) membranes requiring gauge pressures large compared with atmospheric are excluded. In other words, membranes of high curvature or having thick walls may be beyond the description offered here. The criteria derive, in general terms, from two conditions, the first of which is the well-known negative pressure–gradient with respect to volume enclosed, the insufficiency of which is not widely appreciated. The second, which is generally unrecognized, is expressed in various guises according to the peculiar geometry of a particular membrane. It relates fundamentally to the existence of a critical volume of fluid which must be enclosed *within the membrane and the associated tubing and equipment or other reservoir* if instability is to occur once the first condition is met. It is vital that any stability analysis of inflated membranes include for consideration all fluid (air) with which that enclosed in the membrane itself may communicate, particularly any enclosed within a flexible or extensible or adjustable boundary. The general conditions for interconnected inflated lobular membranes are derived in the next Section 2. The derivation is introduced by a discussion of the particular properties of interconnected soap films only in order to emphasize their natural relationship with membranes of more general type. It is in no sense necessary to the argument, to do this.

In the third section the stability of inter-connected inflated spherical rubber membranes is analysed as an illustration of the implications of the general criteria, a direct application of which leads to stability diagrams specific to spheres of a prescribed material. Implicit in these diagrams, however, is the behaviour of nonspherical membranes of like material. New experiments with rubber balloons verify the analytical conclusions.

2. STABILITY CRITERIA FOR MULTI-LOBED MEMBRANES

2.1 *Inter-connected spherical soap films*

Soap films are characterized by a constant surface stress-resultant arising from molecular forces. As a consequence of this, the sum of the principal curvatures of the surface must be constant over the entire soap film, and the inflating pressure must be directly proportional to this sum which generally undergoes an initial increase with increasing size followed by a steady decrease with further increase in size. The requirement of an energy minimum for stable equilibrium leads to the simple condition that the surface area of the soap film must be a minimum for the boundary conditions pertaining and for the volume enclosed.

In a classical experiment with soap films (see, for example, Boys [15] and Smith [16] for illustrated accounts) a small spherical bubble is inter-connected with a larger spherical bubble; the smaller one becomes still smaller while the larger one grows in size, as shown in Fig. 1(a). The explanation is well known and is conventionally presented [Fig. 1(b)], in terms of the equilibrium relation $P = 4T/R$, in which P is the inflating pressure, T is the constant surface stress-resultant for one side of the film and R is the radius of the sphere. The bubbles, initially at points A and B of the hyperbolic characteristic, are inter-connected and air flows from bubble A to bubble B. This representation obscures the fact that there is a pressure maximum at C when the bubble A is hemispherical; and that, therefore, the radius of curvature of the smaller bubble increases as it becomes flatter until equilibrium is attained at D when both bubbles have the same curvature. However, one is practically flat and the other very nearly a complete sphere, the two forming complementary portions of a single sphere, which is clearly the minimal surface area enclosing a given volume. These obscurities can be removed by plotting the pressure against a monotonic quantity such as the height H or the total volume V of the bubbles. In terms of the height H of the crown of a single bubble above the edge ring of common radius R_0 for both bubbles [Fig. 1(a)], the

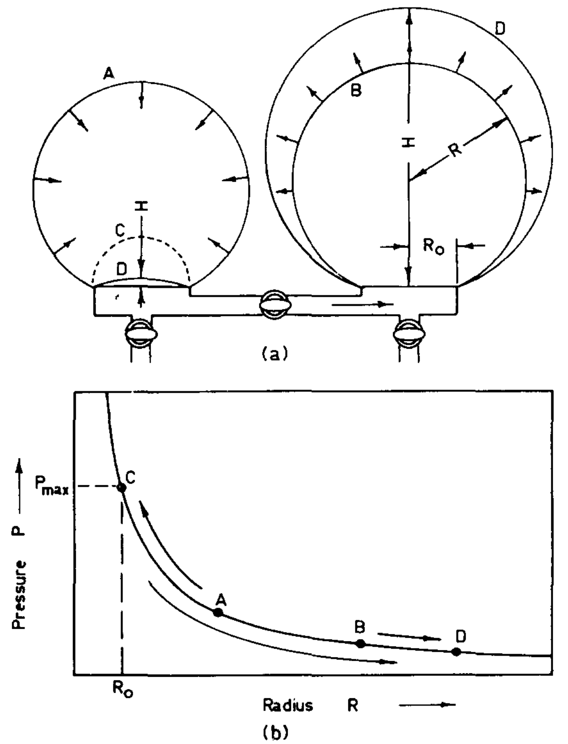

Fig. 1. Instability of inter-connected spherical soap films.

inflating pressure is given by

$$P = \frac{2P_{max}(H/R_0)}{1+(H/R_0)^2} \tag{2.1}$$

where $P_{max} = 4T/R_0$.

The pressure, thus, attains a maximum value (Fig. 2) at $H/R_0 = 1$, as is well-known, when the bubble is precisely hemispherical.

Consider now the possible states of equilibrium of two interconnected bubbles at the same pressure, when the equilibrium condition (2.1) applies equally to both. The bubbles are either of identical volume or not. In the first event, the total volume is twice that of a single bubble; that is, the total reduced volume $V_{red} = V/R_0^3 = (\pi/3)(H/R_0)[3+(H/R_0)^2]$, a relationship which plots in the upper part of Fig. 2 as the curve ABC. In the second event, it is not difficult to show that the total volume is always that of a complete sphere, as already implied, so that, through equation (2.1), the total reduced volume becomes $(\pi/6)\{[1+(H/R_0)^2]/(H/R_0)\}^3$, the curve DBE of Fig. 2. The point B has the co-ordinates $(1, 4\pi/3)$. All possible equilibrium states of two inter-connected bubbles supported on rings of identical radius R_0 are contained in this representation and it is to be observed that for volumes (masses)† sufficiently high, $V_{red} > 4\pi/3$, dual configurations exist: it remains to discriminate the stable from the unstable configurations.

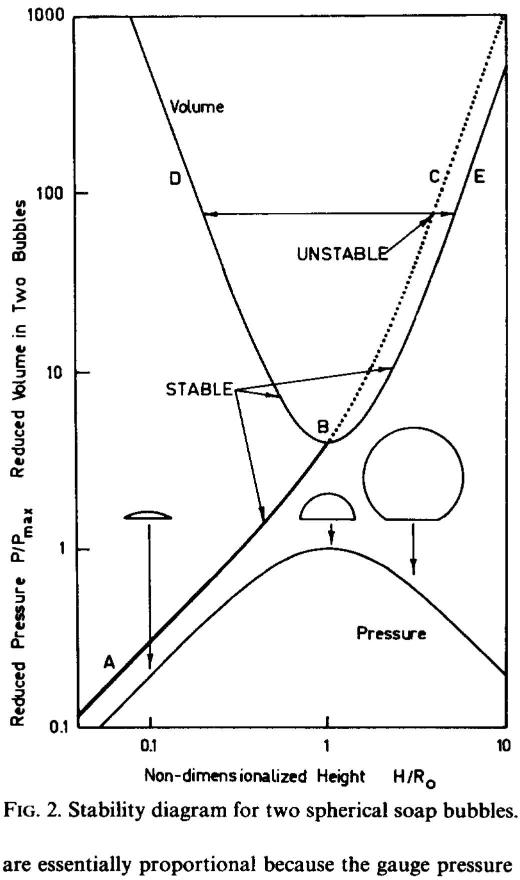

FIG. 2. Stability diagram for two spherical soap bubbles.

† The mass and volume are essentially proportional because the gauge pressure is negligible by comparison with atmospheric pressure.

Now along ABD, $H/R_0 < 1$, it can be shown that $\partial P/\partial V_i$ is positive, V_i being the volume of a single bubble, whereas along BC and BE, $\partial P/\partial V_i$ is always negative. It follows that for two interconnected identical bubbles the equilibrium state is stable for $H/R_0 \leq 1$, along AB, and unstable for $H/R > 1$, along BC; for any transfer of air creates a pressure difference that opposes the transfer in the former event, and accelerates it in the latter event, from the smaller to the larger bubbles. The two bubbles being different in size, similar reasoning shows the equilibrium states, represented by the curve DBE, to be always stable. For any transfer of air creates, in the bubble having $\partial P/\partial V_i > 0$, an opposing pressure difference and in the bubble having $\partial P/\partial V_i < 0$ an accelerating pressure difference, the former influence always dominating the latter, *for spherical bubbles of constant surface stress-resultant*: the sum $\partial P/\partial V_1 + \partial P/\partial V_2$ is, in fact, proportional to $1-[P/P_{max}]^2$ which is always positive, except when $P = P_{max}$ (at the point B of Fig. 2), when it becomes zero. At this point B, each of $\partial P/\partial V_1$ and $\partial P/\partial V_2$ is zero (neither $\partial P/\partial V_1$ nor $\partial P/\partial V_2$ can, by itself, be zero for a *common* pressure) and it becomes necessary to determine the nature of stability by considering the higher derivatives. The second derivatives are, in this case, the same (and actually negative), so that the third derivatives (which are both positive) assure stability.

It is evident, then, that for two spherical membranes characterized by identical *constant* surface stress-resultants and subject to a *common pressure* the nature of the stability is governed by sufficient conditions of the sort:

(i) $\partial P/\partial V_1 > 0$ and $\partial P/\partial V_2 > 0$ stable
(ii) $\partial P/\partial V_i > 0$ and $\partial P/\partial V_j < 0$ stable
(iii) $\partial P/\partial V_1 = 0$ and $\partial P/\partial V_2 = 0$ stable
(iv) $\partial P/\partial V_1 < 0$ and $\partial P/\partial V_2 < 0$ unstable,

where the first and second bubble are denoted by subscripts 1 and 2, respectively, and the last condition (iv) is necessary, as well, for instability.

Examination reveals that the addition of any number of bubbles each less than a hemisphere and at the common pressure of the two given bubbles has no effect on the stability in case (i), while the addition of even one bubble greater than a hemisphere renders case (ii) unstable because it transforms that configuration to essentially that pertaining in case (iv). If a sufficiently large number of bubbles each less than a hemisphere are combined with one larger than a hemisphere so that $\partial P/\partial V_i$ is more negative for the large bubble than $\partial P/\partial V_i$ is positive for the combination of all the smaller ones, the large bubble is now rendered unstable and one of two stable equilibrium forms is adopted. That this is so is established by considering a small amount of air transferred from the larger to the smaller bubbles. The excess pressure is now in the larger bubble and the transfer continues until a stable state is reached in which all the bubbles are the same size and each is less than a hemisphere. If the disturbance is of the opposite form and the excess air is transferred to the larger bubble, the pressure excess is now contained within the smaller bubbles and the larger bubble increases in size: it does not, in fact, burst because no matter how great the (finite) number of small bubbles added $\partial P/\partial V_i$ cannot remain more negative for the larger bubble than $\partial P/\partial V_i$ is positive for the sum of the smaller (nearly flat) bubbles as the volume in the smaller bubbles approaches zero when practically all of the air is in the large bubble. Thus, condition (ii) requires slight modification to cater for a multi-lobed membraneous structure with more than two lobes.

It might be remarked, finally, that the interconnection of two bubbles each having a peculiar constant surface tension T and, therefore, of different material, affords the possibility of more elaborate behaviour. This is true, also, when the boundary radius R_0 differs for each bubble, whether of the same material or not; for then the pressure–height curves are not coincident. The nature of the additional equilibria may be appreciated from the corresponding later discussion of unequal rubber-like membraneous lobes.

Figure 2 can also be interpreted as an unambiguous criterion of the stability of inter-connection of two bubbles initially at differing pressures. Thus, the pressure or height of each is located on the lower pressure curve and projected separately and vertically to intercept the line ABC. The mean of the two volume intercepts determines the total volume enclosed by the two bubbles. And if this mean lies on the part AB of the curve, it defines, by reverse projection, the pressure and height of the stable pair of identical bubbles which result from inter-connection. If, however, the mean intercept lies on the part BC of the curves ABC, the corresponding horizontal intercepts with the curve DBC designate, by projection to the pressure curve, the common pressure and differing heights of the two unequal bubbles that form a stable configuration derived from the initial unequal pair by interconnection. It becomes very clear that the conventional statement concerning the stability of two such inter-connected bubbles is incomplete, for it is restricted to an initial configuration consisting of bubbles each at least hemi-spherical in size and having, therefore, a total volume in excess of that corresponding to the critical state B. For conditions other than this, the smaller bubble enlarges or diminishes while the larger bubble diminishes or enlarges, respectively, according as the volume enclosed is less or greater than this critical volume. In particular, if both bubbles initially form less than a hemisphere the behaviour is the converse of that of two bubbles each greater than a hemisphere: two identical bubbles then co-exist in stable equilibrium.

2.2 *General stability criteria for multi-lobed inflated membranes*

It is to be noted that, in deducing the criteria (i)–(iv) above, the physical argument invoked the special geometric properties of the sphere and the special circumstance of constancy of the surface stress-resultant only in relation to the conditions (ii) and (iii). And this became necessary in order to determine the relative magnitudes of the absolute values of the pressure gradients of each individual bubble. By relaxing these special restrictions on geometry and material an entirely similar argument can be propounded and analogous conditions of stability adduced which apply quite generally to arbitrary arrangements of arbitrarily shaped membraneous lobes of whatever material. The more general criteria, it might be expected, differ only in being more explicit about the relative magnitudes of the absolute values of $\partial P/\partial V_i$ when these exhibit different signs.

Thus, consider n lobes each of arbitrary shape and material of which m lobes encompass a volume V_m and the remainder a volume V_{n-m}. Associate with each of these volumes, respectively, the pressures P_m and P_{n-m} and the small volume changes ΔV_m and ΔV_{n-m} resulting from the transfer of a small mass of air between them. If P be the common pressure obtaining in the equilibrium state prior to the transfer then

$$\begin{aligned} P_m &= P+\frac{\partial P}{\partial V_m}\Delta V_m+\frac{1}{2}\frac{\partial^2 P}{\partial V_m^2}[\Delta V_m]^2+\frac{1}{6}\frac{\partial^3 P}{\partial V_m^3}[\Delta V_m]^3+\dots, \\ P_{n-m} &= P+\frac{\partial P}{\partial V_{n-m}}\Delta V_{n-m}+\frac{1}{2}\frac{\partial^2 P}{\partial V_{n-m}^2}[\Delta V_{n-m}]^2+\frac{1}{6}\frac{\partial^3 P}{\partial V_{n-m}^3}[\Delta V_{n-m}]^3+\dots, \end{aligned} \tag{2.2}$$

and, the system being closed, $\Delta V_m + \Delta V_{n-m} = 0$ relates the volume changes when the distending fluid is essentially incompressible. The pressure differential created by the transfer is, therefore,

$$P_m - P_{n-m} = \Delta V_m \left[\frac{\partial P}{\partial V_m} + \frac{\partial P}{\partial V_{n-m}}\right] + \frac{1}{2}[\Delta V_m]^2 \left[\frac{\partial^2 P}{\partial V_m^2} - \frac{\partial^2 P}{\partial V_{n-m}^2}\right] + \frac{1}{6}[\Delta V_m]^3 \left[\frac{\partial^3 P}{\partial V_m^3} + \frac{\partial^3 P}{\partial V_{n-m}^3}\right] + \ldots$$

which is usefully written

$$\frac{P_m - P_{n-m}}{\Delta V_m} = \left(\frac{\partial P}{\partial V_m} + \frac{\partial P}{\partial V_{n-m}}\right) + \frac{1}{2}\Delta V_m \left(\frac{\partial^2 P}{\partial V_m^2} - \frac{\partial^2 P}{\partial V_{n-m}^2}\right) + \frac{1}{6}\Delta V_m^2 \left(\frac{\partial^3 P}{\partial V_m^3} + \frac{\partial^3 P}{\partial V_{n-m}^3}\right) + \ldots \tag{2.3}$$

Now stability requires a pressure differential which opposes the transfer so that, for stability of the initial equilibrium state, $P_m - P_{n-m} \gtrless 0$ according as the transfer increases or decreases, respectively, the volume V_m; that is, according as $\Delta V_m \lessgtr 0$. Consequently, for *stability* against a perturbation ΔV_m that is so small that $(\Delta V_m)^2, (\Delta V_m)^3, \ldots$ may be neglected, it is necessary that the conditions

$$\frac{\partial P}{\partial V_m} + \frac{\partial P}{\partial V_{n-m}} > 0 \tag{2.4}$$

be satisfied for all possible values of $m = 1, 2, \ldots, n-1$ and, for a given m, for all possible combinations of lobes. These p conditions† are necessary because the failure of (2.4) for just one combination of lobes for some one value of m allows instability. Taken together the conditions (2.4) are *sufficient* to ensure stability, for the class of volume perturbation considered.

In the event that both $\partial P/\partial V_m$ and $\partial P/\partial V_{n-m}$ simultaneously vanish, the inequality of the second derivatives of equation (2.3) ensures instability. And should the second derivatives be equal, the sum of the third derivatives provides the criterion as was the case at the equilibrium bifurcation point for the two spherical bubbles of constant surface tension. It is evident, too, that a sufficient condition for instability is that the sum (2.4) of the two first derivatives be negative, or zero (if metastability is classed as instability), for any single combination of lobes.

It is convenient, in a particular problem, to assess the gradients $\partial P/\partial V_m$ and $\partial P/\partial V_{n-m}$ individually for sign and magnitude. It follows, then, from the criterion (2.4) that a multi-lobed inflated membraneous structure is

1. *stable* if $\partial P/\partial V_m > 0$ and $\partial P/\partial V_{n-m} > 0$ for all and any m;
2. *stable* if $\partial P/\partial V_m > 0$ and $\partial P/\partial V_{n-m} \leq 0$ for r combinations V_m, V_{n-m}, provided that $\partial P/\partial V_m > |\partial P/\partial V_{n-m}|$ for these r combinations, and the condition is satisfied for all remaining volumetric combinations;
3. *unstable* if $\partial P/\partial V_m \geq 0$ and $\partial P/\partial V_{n-m} < 0$ and also $|\partial P/\partial V_{n-m}| > \partial P/\partial V_m$ for some one combination V_m, V_{n-m};
4. *unstable* if $\partial P/\partial V_m < 0$ and $\partial P/\partial V_{n-m} < 0$ for some one combination of lobes for any one value of m.

† Here $p = \sum_{m=1}^{s} {}_nC_m$, where $s = n/2$ and $(n-1)/2$ for n even and odd, respectively. This gives, for n odd, $p = 2^{n-1} - 1$ or $p = 0, 3, 15, 63, \ldots$.

For the case of identically zero pressure gradients for all lobes in some one combination or more, the first derivatives are to be replaced by higher derivatives as earlier indicated.

The condition in case 3 for which $\partial P/\partial V_m = 0$ and $\partial P/\partial V_{n-m} < 0$ caters for the case of an individually-stable inflated membrane with a negative pressure gradient communicating with an inextensible vessel of infinite capacity filled with air at the same pressure: that is, for the case of *constant pressure* instability. For, if some air be transferred from the membrane to the tank, the pressure excess will be in the membrane, and hence the transfer of air will continue. Likewise, if some air be transferred to the membrane, the pressure excess will now be in the reservoir and the situation is again unstable. This explains a phenomenon noted by Searle [17] that an initially-stable soap-film cylinder for which $\pi \leq l/r \leq 2\pi$ becomes unstable when communicating with an infinite reservoir, while an isolated cylinder is stable until $l/r \geq 2\pi$. The explanation of the different critical lengths lies in the fact [14] that for a cylindrical soap film supported on two parallel circular rings of radius r separated by a distance l, $\partial P/\partial V > 0$ for $l/r < \pi$, $\partial P/\partial V = 0$ for $l/r = \pi$ and $\partial P/\partial V < 0$ for $l/r > \pi$. Therefore, one cylinder of $l/r > \pi$ inter-connected with an infinite reservoir is unstable in accordance with criterion 3, while an isolated cylinder of $l/r \geq 2\pi$ is unstable because it represents two identical cylinders in combination, each of $l/r \geq \pi$, which are unstable according to criterion 4.

In contrast with soap-films, rubber-like membranes are characterized by principal surface stress resultants which increase with increasing extension—see, for example, equations (2.13) and (2.14) of Ref. [2]. Furthermore, the pressure almost invariably initially increases to a maximum at quite a small increase in size (less than 50 per cent) and decreases to a minimum (at about three to four times the size) after which the limited extensibility of the long-chain rubber molecules results in a pressure increase notwithstanding the steady decreases in principal curvatures. The condition of minimal surface area no longer obtains. This final pressure increase for rubber-like membranes increases the number of stable configurations for inter-connected rubber membranes over that which holds for soap films, but otherwise their behaviour is very similar, and the criterion (2.4) and the conditions 1–4 cater equally for both; and, indeed, for all membranes regardless of composition.

3. STABILITY OF INFLATED INTER-CONNECTED SPHERICAL RUBBER MEMBRANES

It has already been hinted that the modes of behaviour possible of membranes of different materials such as the rubber-like long-chain polymers, are more diverse than those already discussed, yet still consistent with the general criteria. And this diversity is now to be elucidated by a consideration, first, of spherical rubber membranes. By this means, all geometric criteria such as those governing the stability of the soap-film cylinder, unduloid, nodoid and catenoid (as outlined in the Appendix) can be excluded, and only material effects considered.

Spherical rubber balloons have been studied in great detail elsewhere [1], and in quantitative experiments on isolated balloons no instability was observed at any stage. In applying the stability criteria above to two inter-connected balloons, it is necessary to take as a starting point the pressure–mass relation for a single sphere. As before, the enclosed mass is again directly proportional to the volume of a balloon and the pressure and volume, in lieu of mass, can be inter-related through the deformation of the membrane. Using the exponential–hyperbolic elasticity parameters [1] to represent the material properties, the non-dimensionalized pressure and volume may be expressed in terms of the circumferential extension ratio λ in the form

$$P' = \frac{PA}{2hG} = 2\left[\frac{1}{\lambda}-\frac{1}{\lambda^7}\right]\left[e^{k_1 E_1^2}+\lambda^2\frac{k_2}{I_2}\right], \tag{3.1}$$

$$V' = 3V/4\pi A^3 = \lambda^3, \tag{3.2}$$

where P is the actual inflating gauge pressure, A is the initial radius of the membrane (balloon), h is the initial thickness, G, k_1 and k_2 are the finite-elasticity constants, V is the volume and the strain invariants E_1 and I_2 are expressed here as functions of the extension ratio λ by the relations

$$E_1 = 2\lambda^2+(1/\lambda^4)-3, \tag{3.3}$$

$$I_2 = \lambda^4+(2/\lambda^2). \tag{3.4}$$

This pressure–volume relation for a single sphere (which is substantiated by quantitative experiments elsewhere [1]) is plotted in the lower part of Fig. 3 for typical values of the finite-elasticity constants ($k_1 = 0{\cdot}0002, k_2 = 0{\cdot}5$). The existence of a pressure maximum followed by a minimum, which is characteristic of nearly all inflated rubber membranes, is

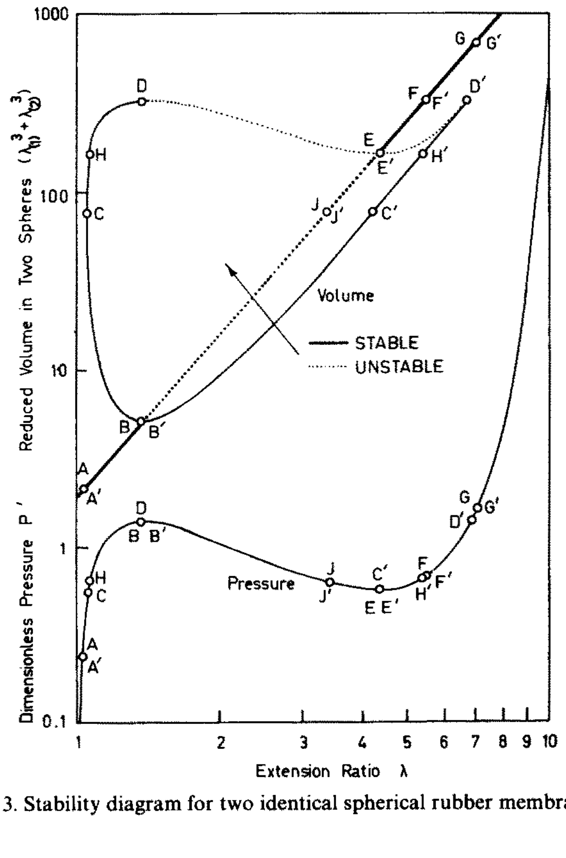

FIG. 3. Stability diagram for two identical spherical rubber membranes.

evident. This contrasts with the pressure–volume relation for spherical soap films (Fig. 2), which exhibits only the pressure maximum. And it is this property which allows a greater number of stable equilibrium configurations for the rubber membranes. It cannot be overemphasized that a realistic analysis and understanding of their diverse behaviour can only be attained by the proper representation of the elastic properties of the membrane.† Some of the consequences of mis-representing physical fact, in this regard, become apparent in discussion elsewhere on cylindrical membranes.

Two initially identical spheres, each characterized by the same pressure–extension curve exhibited in the lower part of Fig. 3, which are inflated to enclose a given total volume possess stable configurations upon interconnection determined solely by that total volume and the criteria 1–4 of the previous section. Thus, in the upper part of Fig. 3 are shown for a prescribed total volume, the available configurations of the two spheres whether identical (along AG) or not (along $BDED'B'$). The nature of the stability, at any volume, is determined directly from the criteria 1–4 with the results indicated in Fig. 3.

The significant differences in behaviour between that of the rubber membranes and the soap films lie in the stable co-existence of two large rubber spherical membranes in the positive pressure-gradient region EG and beyond, a region not found in the soap film pressure characteristic. As a consequence of this, the rubber membranes exhibit also a bistable region in which two identical inter-connected balloons in stable co-existence between regions E–F and E'–F', respectively, can be popped-through to an alternative stable state. In this, one very small balloon persists in the region D–H and the other balloon, larger than before, in the region D'–H'. As with the soap-bubble membranes, two inter-connected spheres in the negative pressure-gradient region B–E (or B'–E') represent an unstable combination; one bubble decreases in size to become smaller than at the peak pressure, while the other one increases. In the region B'–J' the larger balloon remains in the negative pressure-gradient region, while from J' to E' it passes to the second positive pressure-gradient region C'–D'.

Inasmuch as the stability curves are determined essentially by the nature of the pressure-deformation characteristic, the same qualitative behaviour can be attributed to membranes having like characteristics. In particular, a qualitatively identical stability chart can be deduced for the inflation of inter-connected uniform flat circular rubber membranes, which exhibit the same sort of inflation characteristic [2].

Figure 3 defines, also, the behaviour of two inter-connected identical balloons during simultaneous inflation. Until the peak pressure is reached the two spherical balloons remain identical (along AB) but, thereafter, one decreases in size from B to C while the other increases in size along the negative pressure-gradient region, from B' to C'. The combination is stable at all stages, since the stable equilibrium is determined uniquely by the volume of air enclosed. As the large balloon passes through the pressure minimum at C', the smaller balloon starts increasing in size again until in due course the pressure maximum is attained at D. Upon further inflation the balloons undergo a snap-through transformation into two identical balloons at F and F'. If air be withdrawn at this stage, the two balloons together diminish in size but remain identical, until the pressure minimum is reached (E–E'); and if further air be extracted, a snap-through transition to the different states H and H' occurs. Alternatively, both balloons being at (or beyond) the pressure minimum (E–E'), they increase

† The influence of the values of the finite-elasticity constants k_1 and k_2 on the value of the extension ratio of the maximum and minimum pressures is discussed elsewhere [1].

in size together as air is admitted, remaining identical until failure. Naturally, at any stage in the intervals B–C–D and B'–C'–D', the sizes of the respective balloons can be interchanged to reverse the configuration, which remains stable.

The bi-stable configuration is readily observed experimentally, as implied by Fig. 4, but the actual simultaneous inflation of two balloons reveals many other phenomena. These can be explained in terms of differing initial thicknesses (or sizes) of the two balloons which means that one reaches its peak pressure prior to the other. Then, the Fig. 3 takes on the appearance of Fig. 5, in which one of the balloons (designated as sphere 2 and represented by the primed lettering and associated with the subscript 2), is half as thick, or twice as large, as the other (designated as sphere 1). Its inflating pressure is always one half that of the other for any common extension-ratio, as shown in the pressure characteristic in the lower part of the figure. When simultaneously inflated,† both spheres initially increase in size, sphere 1 from A to B and sphere 2 from A' to B'. When sphere 2 reaches the pressure maximum at B', continuing injection of air causes sphere 2 to grow steadily in size from B' to C', while sphere 1 decreases from B to C. Thereafter, sphere 1 increases in size again from C to D while sphere 2 follows the second positive pressure-gradient from C' to D'. Further

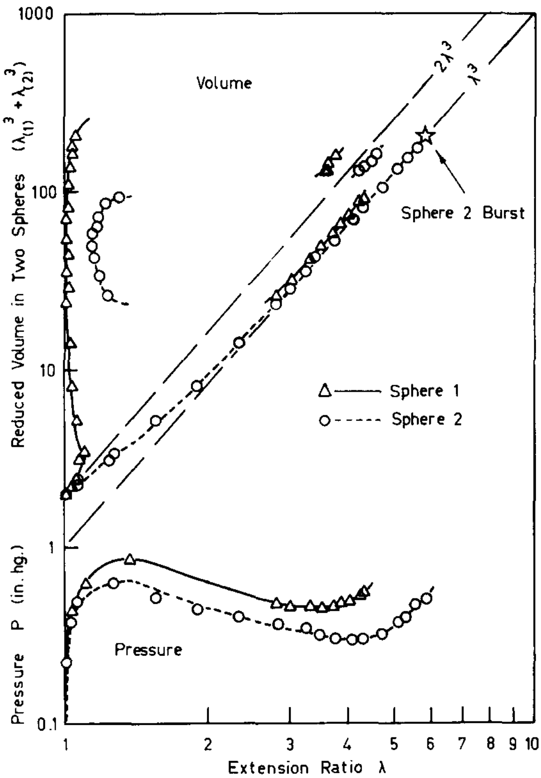

FIG. 4. Stable combinations of two rubber spherical membranes—experiment.

† The following description explains the phenomenon of Pollock and Boshes [23] who remark on the influence of the pressure–volume gradient during the distension of membranes. "This is seen in the case of a balloon partially constricted at its upper one-third by a ligature. When a head of pressure is put on the contents of such a balloon, the part with the larger area distends until it is rigid and large; the smaller part seems softer and is but little distended."

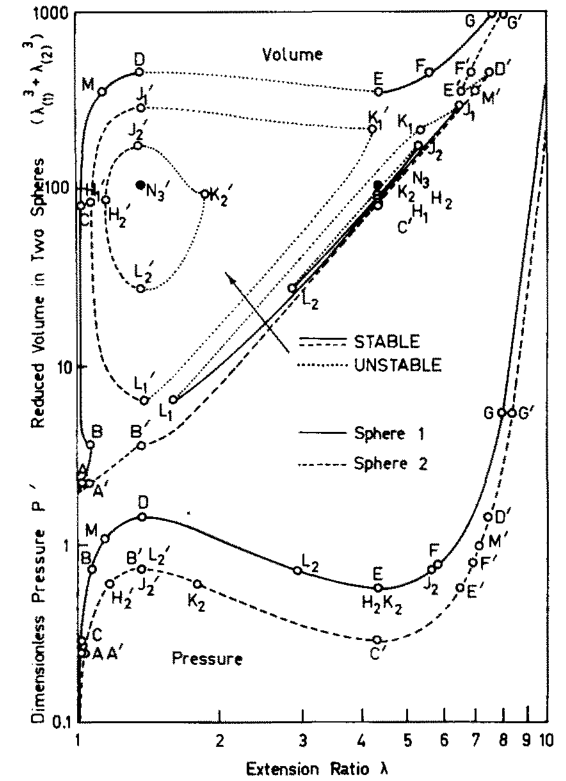

FIG. 5. Stability diagrams for two spherical rubber membranes.

admission of air at this stage causes a snap-through transition to F and F', respectively, in which state the initially thinner (or larger) balloon 2 is at a higher extension-ratio than is balloon 1 (if initially larger it appears larger still in proportion). If air be withdrawn at the states F and F' the balloons decrease in size until the states E and D' are reached, at which sphere 1 has reached its pressure minimum. Further removal of air causes a snap-through transition to M and M'. As with the identical balloons, introduction of more air beyond E–E' causes both balloons to grow steadily in size.

Provided that the pressure maximum of the thinner (or larger) balloon is greater than the pressure minimum of the other balloon, the direct inter-change of the states B–C–D and B'–C'–D' which could be achieved from identical balloons is now replaced by an indirect inter-change to a further equilibrium combination which is isolated from those discussed in the paragraph above. If the two balloons are independently brought to their common negative pressure-gradient region L–K and L'–K' respectively and then inter-connected, they do not snap through to any of the stable states above. Instead, sphere 2 decreases in size to the region below L' extending slightly beyond H', while sphere 1 increases in size beyond L; and depending on how close the original states are to K, remain in the negative pressure-gradient region or may increase to just past K. Further air added to this combination maintains equilibrium with sphere 1 on the second positive pressure-gradient H–J and sphere 2 in the region H'–J' until the second sphere reaches its peak pressure at J'. A snap-through transition then occurs in which sphere 1 greatly decreases in size from more than that at the pressure minimum to less than that at the pressure maximum, while sphere 2 increases in size from that at the pressure maximum to more than that at the pressure minimum, which is seen from the upper curve in Fig. 4 to be the only stable combination for that particular enclosed volume. Although the addition of excess air to the discrete loops L–H–J–K and L'–H'–J'–K' will cause a snap-through transition to the outer envelopes B–C–D and B'–C'–D', the converse cannot be achieved without additional constraints to suppress the normal decrease in size of both balloons together. The upper diagram in Fig. 5 corresponds accurately to the lower one for the points designated by subscript 2, while the points with subscript 1 refer to a balloon with only a slightly lower peak pressure than sphere 1. Points N_3 and N'_3 indicate the limit to which the loops L–H–J–K reduce when the maximum pressure of sphere 2 coincides with the minimum pressure of sphere 1. The existence of this discrete stability loop can be demonstrated on the rig on which the experimental observations recorded in Fig. 4 were made.

In this rig each of up to three balloons can be individually inflated or deflated while the remainder is isolated; and any two or more balloons can be inter-connected. The pressure in each individual balloon is displayed on an attached U-tube manometer. This rig was used to verify qualitatively all the stability criteria developed in Section 2 for multi-lobed inflated membraneous structures. The U-tube manometers facilitated bringing each balloon to the same pressure before opening the valve(s) between them. Even the discrete stability loop in Figs. 4 and 5 for spherical balloons of a different thickness (or size) was found to obey the criteria 1–4.

Figure 6, based on Fig. 3, is a theoretically-derived stability chart for three identical inter-connected spherical rubber balloons. Here a small tri-stable region is evident as well as two bi-stable ones. The numbers on the lines refer to the number (1, 2 or 3) of balloons in that state. This behaviour was verified qualitatively on the rig above and, as predicted by the criteria, three identical balloons can co-exist on either of the two positive pressure-gradient regions ab, fg.

Upon simultaneous inflation from zero pressure several phenomena are possible as the total enclosed volume becomes large. Beyond and above the bifurcation point b, one very large sphere proceeds to c along the right-hand branch bc while two small and identical spheres proceed to c along the left-hand branch bc. An appropriate disturbance results in snap-through of all three spheres to the stable branch fg when they all assume an identical size, slightly smaller than the original large sphere. In such a simultaneous snap-through of the two smaller spheres the system behaves essentially as a 2-lobed one. A more likely event is the transition of the single large sphere from the right-hand point c to the nearer and adjacent stable branch de in which it suffers a small diminution in size. Of the two small and identical spheres at the left-hand points c one enlarges extremely to equalize in size the first sphere while the other decreases slightly in size to stabilize on the left-hand branch de. It requires, then, further injection of air to cause this last sphere finally to snap-through with a major change in size to join the other equal spheres on the branch fg, when they all become identical. Superficially, then, the gross appearance is a consecutive snap-through of the two smaller spheres to match in size the first fully inflated sphere. This behaviour is consistent with the behaviour of multi-lobed party balloons in which the balloon inflates progressively one lobe at a time, with never more than one lobe at a time in the negative pressure-gradient region. The thickness variation in these balloons is such that the first lobe usually, but not always, inflates first and the end lobe last, as depicted in Fig. 7.

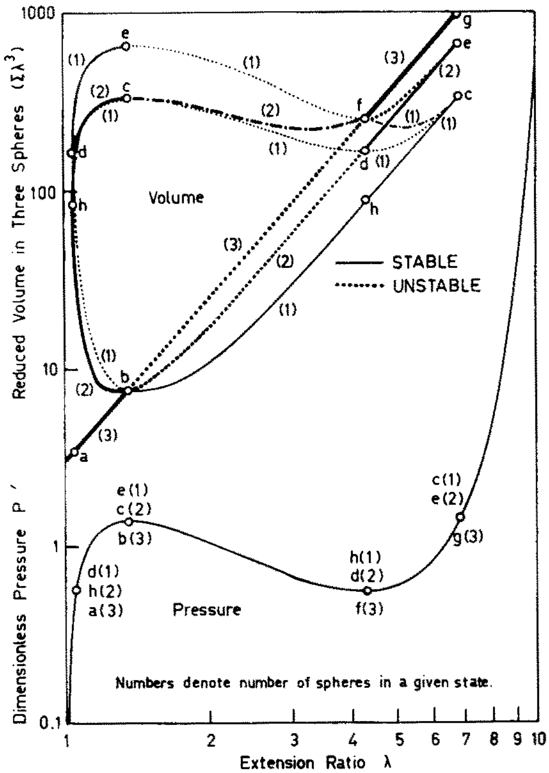

FIG. 6. Stability diagram for three identical spherical rubber membranes.

4. CONCLUSION

The investigations above, both theoretical and experimental, of inter-connected rubber spheres and of inter-connected soap films confirm, as special examples, that the stability criteria proposed quite generally and separately in Section 2 are realistic; and that in each case *two conditions* are required to discriminate between stability and instability. The first of these is a negative pressure-gradient with respect to volume, which is well-known, but the insufficiency of which is not so widely appreciated. The other is a geometric condition which adopts various guises according to the specific situation under investigation. Basically it represents a critical mass (volume) which must be enclosed within a system if instability is to occur once the first condition is met.

For example, two inter-connected spherical balloons are in stable equilibrium if one is in the negative pressure-gradient region and the other is in the first (smaller-size) positive pressure-gradient region, even though the pressure-gradient for the *entire* system is *negative*. But, when more air is added, so that *both* balloons are in the negative pressure-gradient region, the combination is unstable, remaining so while one balloon is in the negative pressure-gradient region and the other is inflated into the second (larger-size) positive pressure-gradient region. With further increase in the amount of air enclosed the instability is removed as both balloons reach the second positive pressure-gradient region.

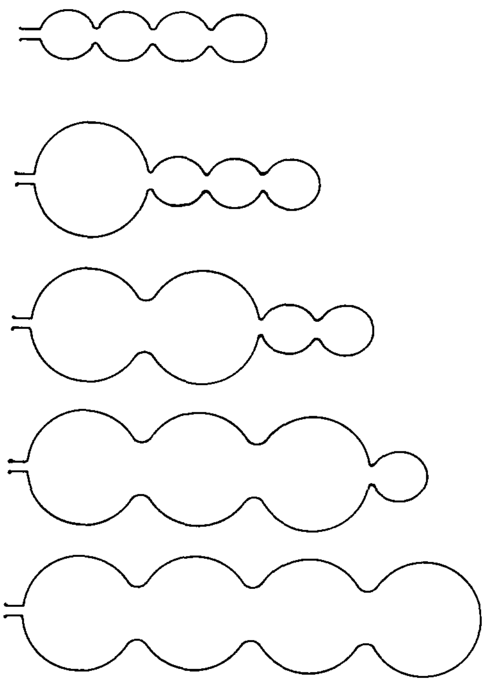

FIG. 7. Inflation of multi-lobed balloon.

Most inflated rubber membranes possess a characteristic pressure–inflation relation with two positive pressure-gradients with respect to volume separated by a region of negative pressure-gradient, there being, consequently, a pressure maximum followed by a pressure minimum. This characteristic is sufficient to ensure that any two inter-connected balloons possess a bi-stable region in which two identical balloons near the lower size limit of the second positive pressure-gradient can be popped-through to an alternative stable combination of one small balloon on the first positive pressure-gradient region with the other remaining in the second, and vice versa. Three inter-connected balloons may have a tri-stable as well as two bi-stable regions.

The comparison of previous analyses of such topics as are discussed here with the present one has emphasized the need for maintaining a close liaison between the theoretical representation of the problem and the actual physical situation. In particular, in invoking the stability criteria derived here, it is important to represent the elastic properties of the membrane in a realistic manner if the diverse stability phenomena are to be accurately predicted, even qualitatively. Either the exponential–hyperbolic elasticity parameters [1] or the Carmichael and Holdaway [18] stress function are appropriate for rubber-like materials.

REFERENCES

[1] L. J. HART-SMITH, Elasticity parameters for finite deformation or rubber-like materials. *Z. angew. Math. Phys.* **17**, 608 (1966).

[2] L. J. HART-SMITH and J. D. C. CRISP, Large elastic deformations of thin rubber membranes. *Int. J. Engng Sci.* **5**, 1 (1967).

[3] A. MALLOCK, Note on the instability of india-rubber tubes and bellows when distended by fluid pressure. *Proc. R. Soc.* **49**, 458 (1891).
[4] W. A. OSBORNE and W. SUTHERLAND, The elasticity of rubber balloons and hollow viscera. *Proc. R. Soc.* **B81**, 485 (1909).
[5] R. KUBO, Large elastic deformation of rubber. *J. phys. Soc. Japan* **3**, 312 (1948).
[6] M. LEVINSON, The application of the principle of stationary potential energy to some problems in finite elasticity. *J. appl. Mech.* **32**, 656 (1965).
[7] W. JOHNSON and P. D. SODEN, The discharge characteristics of confined rubber cylinders. *Int. J. mech. Sci.* **8**, 213 (1966).
[8] M. LEVINSON, The Finite Strain Behaviour of Incompressible Hollow Cylinders and Spheres under Internal Pressure, Cal. Inst. Tech. CALCIT SM 62–48 (1962).
[9] A. C. BURTON, On the physical equilibrium of small blood vessels. *Am. J. Physiol.* **164 (1)**, 319 (1951).
[10] A. C. BURTON, Relation of structure to function of the tissues of the wall of blood vessels. *Physiol. Rev.* **34 (4)**, 619 (1954).
[11] Y. G. PANOVKO and I. I. GUBANOVA, *Stability of Oscillations of Elastic Systems*, Translated from Russian by C. V. LARRICK. Consultants Bureau (1965).
[12] A. R. RZHANITSYN, *Stability of Elastic Systems*, No. 24. Moscow (1955).
[13] A. H. CORNELIUSSEN and R. T. SHIELD, Finite deformation of elastic membranes with application to the stability of an inflated and extended tube. *Archs ration. Mech. Analysis* **7**, 273 (1961).
[14] J. PLATEAU, *Statique Expérimentale et Théorique des Liquides soumis aux seules Forces Moléculaires*, pp. 291–293. Gauthier-Villars (1873).
[15] C. V. BOYS, *Soap Bubbles*. Dover (1959). (First published 1911).
[16] C. J. SMITH, *General Properties of Matter*, 2nd edition, pp. 457–461. Arnold (1956).
[17] C. F. C. SEARLE, *Experimental Physics*, p. 150 (1950); V. H. L. SEARLE, *General Properties of Matter*, 5th edition, pp. 166–173. Arnold (1957).
[18] A. J. CARMICHAEL and H. W. HOLDAWAY, Phenomenological elastomechanical behaviour of rubber over wide ranges of strain. *J. appl. Phys.* **32**, 159 (1961).
[19] J. C. MAXWELL, Capillary Action, in *Encyclopaedia Britannica*, 9th edition, p. 56 (1875–1889); or *Scientific Papers*, pp. 541–596. Dover (1965).
[20] C. DELAUNAY, Sur la surface de révolution dont la courbure moyenne est constante. *J. Math. pures appl.* **6**, 309 (1841).
[21] D'ARCY W. THOMPSON, *On Growth and Form*, Cambridge University Press (1961). Abridged Edition edited by J. T. BONNER. (First published in 1917.)
[22] LORD RAYLEIGH, On the instability of fluid surfaces. *Phil. Mag.* **34**, 177 (1892); or *Scientific Papers*, Vol. 3, pp. 594–596. Dover (1964).
[23] L. J. POLLOCK and B. BOSHES, Cerebrospinal fluid pressure. *Archs Neurol. Psychiat., Lond.* **36**, 931–974 (1936).

APPENDIX

The stability of soap-film membranes

Soap films are characterized by a constant surface stress-resultant due to the surface tension arising from molecular forces. They have been studied comprehensively in the past, and such an examination is given by Maxwell [19]. For the present purposes it is appropriate to review only the axi-symmetric forms and the conditions governing their stability. This review serves primarily to show how the general stability criteria 1–4 developed in Section 2.2 apply to soap films.

Plateau [14] was the first to show that there are only six surfaces of revolution which an axi-symmetric soap-film can adopt and yet retain equilibrium. These are the plane, the cylinder, the sphere, the catenoid, the unduloid and the nodoid, which are illustrated in Fig. 8.†

† The mathematical inter-relation of these surfaces is expressed by the fact first shown by Delaunay [20] that the plane curves by whose revolution they are generated are themselves generated as "roulettes" of the conic sections. More recent expositions of this inter-relation are given by Thompson [21] and by Maxwell [19].

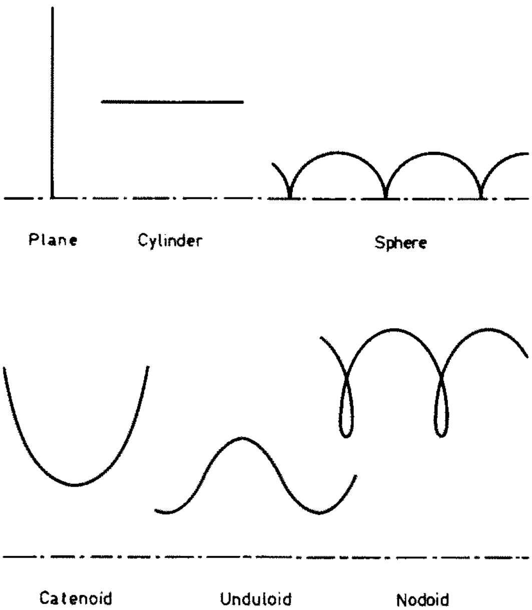

FIG. 8. Plateau's axi-symmetric soap films.

It is noteworthy that of these surfaces only the sphere is a closed surface and capable of existence in isolation, and that the plane and catenoid can exist only in the absence of a pressure differential. Plateau discovered also that, with the exception of the plane and the sphere (or any portion thereof), these surfaces are in complete equilibrium only within certain geometric limits. That is, their stability or instability is determined by their proportions. The symmetry of the perfectly-stable surfaces ensures that any small disturbance readjusts itself leaving the plane or spherical surface whole, provided that it does not communicate with any other volumes of air in the manner of the inter-connected spheres of Section 2. On the other hand, in the remaining configurations any disturbance once set up will be propagated with increasing amplitude.

A well-known instability phenomenon associated with soap films is the disintegration of an isolated cylinder whose length-to-diameter ratio exceeds the critical value π, established by Plateau [14, p. 293]. By considering the surface area of the deformed soapfilm, Plateau established that, for shorter cylinders, a bulge without volume change is associated with a pressure increase, the lower pressure region being the neck (or waist). Such short cylinders are thus stable. The unstable long cylinder is transformed into an increasingly severe unduloid-like surface and eventually disintegrates when the waist has vanished. (The liquid cylindrical jet studied by Rayleigh [22], of which the behaviour is governed by much the same equilibrium equations, breaks up into a series of alternate large and small spheres.) It is apparent, by analogy, that the critical length of the unduloid is one wavelength. Maxwell [19] has shown that the catenoid is stable only when the portion considered is such that the tangents to the catenary at its extremities intersect before they reach the directrix (axis of revolution), except when each end of the catenoid is sealed so as to maintain constant the volume enclosed. There are then two stable catenoids for each set of boundary conditions.

The uniform excess pressure on the nodoid is everywhere on the concave side of the curve, so only isolated portions of it can be realized; and the question of instability cannot therefore arise until one complete wavelength is achieved, whereupon the inner loop closes upon itself and may be removed, in theory, as an inflated doughnut-like membrane. The outer portion that remains is less than a complete wavelength and may be re-inflated. The transition from one of these axi-symmetric soap-films to another is depicted theoretically in Fig. 9 in which the respective diagrams are arranged in ascending order of volume enclosed, proceeding across and then down the page. This sequence does not appear to have been examined before.

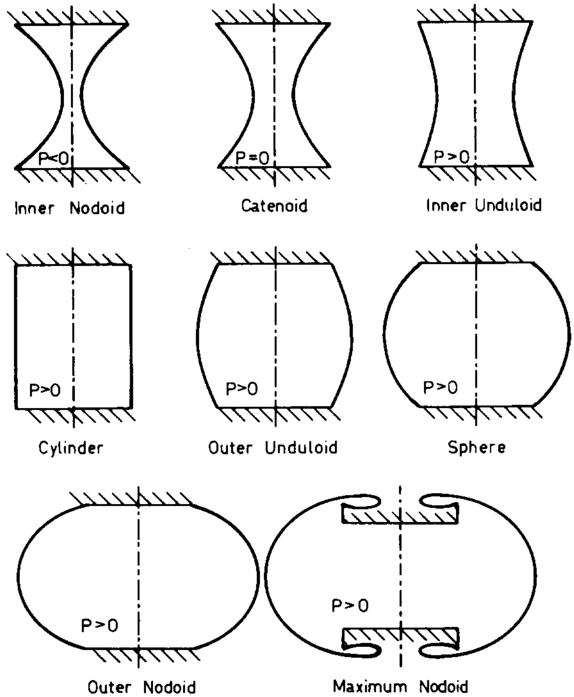

FIG. 9. Axi-symmetric soap films in order of increasing volume.

(*Received* 8 *May* 1967; *revised* 14 *July* 1970)

Абстракт—Исследуется, теоретически, устойчивость надутой мембранной сферы из резцноподобного материала. Критерия устойчивости подтверждаются зкспериментально. Определяется, что отрицательный градиент обьема давления, дает критическое условие для неустойчивости. Другое условие вытекает из геометрических рассуждений. Оно, обыкновенно, заключает критический обьем, заключенный внутри целой системы. Эти два условия определяют, совместно, устойчивость для какой либо наполненной воздухом мембранной конструкции с многими диафрагмами, не обрашая внимания на материал.Представляются состояния равновесия для двух или трех, соединённых с собой, сферических воздушных шаров, которые могут быть одинаковы или нет, совместно с соответствуюшими графиками устойчивости. Дается также история каждой конфигурации. Описывается область исследования многоустойчивых состояний, что подтверждаются эксперименталвно. Приводятся специфичные результаты для резиноподобных материалов, описанных с помошью экспотенциально-гиперболических параметров упругости, выведенных где-нибудь в другом месте в работе [1]. Исследуются несоотвествия других теорий [3–11], либо для сравнения упругих решений, либо для пренебрежения одним из двух условий устойчивости.